Sorting by Transpositions is Difficult

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Abstract. In comparative genomics, a transposition is an operation that exchanges two consecutive sequences of genes in a genome. The transposition distance, that is, the minimum number of transpositions needed to transform a genome into another, is, according to numerous studies, a relevant evolutionary distance. The problem of computing this distance when genomes are represented by permutations, called the SORTING BY TRANSPOSITIONS problem, has been introduced by Bafna and Pevzner [3] in 1995. It has naturally been the focus of a number of studies, but the computational complexity of this problem has remained undetermined for 15 years.

In this paper, we answer this long-standing open question by proving that the SORTING BY TRANSPOSITIONS problem is NP-hard. As a corollary of our result, we also prove that the following problem [8] is NP-hard: given a permutation π , is it possible to sort π using $d_b(\pi)/3$ permutations, where $d_b(\pi)$ is the number of breakpoints of π ?

Introduction

Along with reversals, transpositions are one of the most elementary large-scale operations that can affect a genome. A transposition consists in swapping two consecutive sequences of genes or, equivalently, in moving a sequence of genes from one place to another in the genome. The transposition distance between two genomes is the minimum number of such operations that are needed to transform one genome into the other. Computing this distance is a challenge in comparative genomics, since it gives a maximum parsimony evolution scenario between the two genomes.

The Sorting by Transpositions problem is the problem of computing the transposition distance between genomes represented by permutations. Since its introduction by Bafna and Pevzner [3, 4], the complexity class of this problem has never been established. Hence a number of studies [4, 8, 15, 17, 12, 5, 14] aim at designing approximation algorithms or heuristics, the best known fixed-ratio algorithm being a 1.375-approximation [12]. Other works [16, 8, 13, 19, 12, 5] aim at computing bounds on the transposition distance of a permutation. Studies have also been devoted to variants of this problem, by considering, for example, prefix transpositions [11, 20, 7] (in which one of the blocks is a prefix of the sequence), or distance between strings [9, 10, 23, 22, 18] (where multiple occurrences of each element are allowed in the sequences), possibly with weighted or prefix transpositions [21, 6, 1, 2, 7].

$$\pi = (\pi_0 \pi_1 \dots \pi_{i-1} \underline{\pi_i \dots \pi_{j-1}} \underline{\pi_j \dots \pi_{k-1}} \underline{\pi_k \dots \pi_n})$$

$$\pi \circ \tau_{i,j,k} = (\pi_0 \pi_1 \dots \pi_{i-1} \underline{\pi_j \dots \pi_{k-1}} \underline{\pi_i \dots \pi_{j-1}} \underline{\pi_k \dots \pi_n})$$

Figure 1: Representation of a transposition $\tau_{i,j,k}$, with $0 < i < j < k \le n$.

In this paper, we address the long-standing issue of determining the complexity class of the SORTING BY TRANSPOSITIONS problem, by giving a polynomial time reduction from SAT, thus proving the NP-hardness of this problem. Our reduction is based on the study of transpositions removing three breakpoints. A corollary of our result is the NP-hardness of the following problem, introduced by [8]: given a permutation π , is it possible to sort π using $d_b(\pi)/3$ permutations, where $d_b(\pi)$ is the number of breakpoints of π ?

1 Preliminaries

1.1 Transpositions and Breakpoints

In this paper, n denotes a positive integer. Let $[a;b] = \{x \in \mathbb{N} \mid a \leq x \leq b\}$, and Id_n be the identity permutation over [0;n]. We consider only permutations of [0;n] such that 0 and n are fixed-points. Given a word $u_1 u_2 \ldots u_l$, a subword is a subsequence $u_{p_1} u_{p_2} \ldots u_{p_{l'}}$, where $1 \leq p_1 < p_2 < \ldots < p_{l'} \leq l$. A factor is a subsequence of contiguous elements, i.e. a subword with $p_{k+1} = p_k + 1$ for every $k \in [1; l'-1]$.

A transposition is an operation that exchanges two consecutive factors of a sequence. As we only work with permutations, it is defined as a permutation $\tau_{i,j,k}$, which, once composed to a permutation π , realise this operation (see Figure 1). The transposition $\tau_{i,j,k}$ is formally defined as follows.

Definition 1 (Transposition). Given three integers i, j, k such that $0 < i < j < k \le n$, the transposition $\tau_{i,j,k}$ over [0; n] is the following permutation (we write q(j) = k + i - j):

Note that the inverse function of $\tau_{i,j,k}$ is also a transposition. More precisely, $\tau_{i,j,k}^{-1} = \tau_{i,q(j),k}$. The following two properties directly follow from the definition of a transposition:

Property 1. Let $\tau = \tau_{i,j,k}$ be a transposition, q(j) = k + i - j, and $u, v \in [0; n]$ be two integers such that u < v. Then:

$$\tau(u) > \tau(v) \iff i \le u < q(j) \le v < k$$

$$\tau^{-1}(u) > \tau^{-1}(v) \iff i \le u < j \le v < k$$

Property 2. Let τ be the transposition $\tau = \tau_{i,j,k}$, and write q(j) = k + i - j. For all $x \in [1; n]$, the values of $\tau(x-1)$ and $\tau^{-1}(x-1)$ are the following:

$$\forall x \notin \{i, q(j), k\}, \quad \tau(x-1) = \tau(x) - 1$$

$$\forall x \notin \{i, j, k\}, \quad \tau^{-1}(x-1) = \tau^{-1}(x) - 1$$

$$\tau(i-1) = \tau(q(j)) - 1 \quad \tau^{-1}(i-1) = \tau^{-1}(j) - 1$$

$$\tau(q(j)-1) = \tau(k) - 1 \quad \tau^{-1}(j-1) = \tau^{-1}(k) - 1$$

$$\tau(k-1) = \tau(i) - 1 \quad \tau^{-1}(k-1) = \tau^{-1}(i) - 1$$

Definition 2 (Breakpoints). Let π be a permutation of [0; n]. If $x \in [1; n]$ is an integer such that $\pi(x-1) = \pi(x) - 1$, then (x-1,x) is an adjacency of π , otherwise it is a breakpoint. We write $d_b(\pi)$ the number of breakpoints of π .

The following property yields that the number of breakpoints of a permutation can be reduced by at most 3 when a transposition is applied:

Property 3. Let π be a permutation and $\tau = \tau_{i,j,k}$ be a transposition (with $0 < i < j < k \le n$). Then, for all $x \in [1; n] - \{i, j, k\}$,

$$(x-1,x)$$
 is an adjacency of $\pi \Leftrightarrow (\tau^{-1}(x)-1,\tau^{-1}(x))$ is an adjacency of $\pi \circ \tau$.

Overall, we have $d_b(\pi \circ \tau) \geq d_b(\pi) - 3$.

Proof. For all $x \in [1; n] - \{i, j, k\}$, we have:

$$(x-1,x)$$
 adjacency of $\pi \Leftrightarrow \pi(x-1) = \pi(x) - 1$
 $\Leftrightarrow \pi(\tau(\tau^{-1}(x-1))) = \pi(\tau(\tau^{-1}(x))) - 1$
 $\Leftrightarrow \pi \circ \tau(\tau^{-1}(x) - 1) = \pi \circ \tau(\tau^{-1}(x)) - 1$ by Prop. 2
 $\Leftrightarrow (\tau^{-1}(x) - 1, \tau^{-1}(x))$ adjacency of $\pi \circ \tau$.

1.2 Transposition distance

The transposition distance of a permutation is the minimum number of transpositions needed to transform it into the identity. A formal definition is the following:

Definition 3 (Transposition distance). Let π be a permutation of [0; n]. The transposition distance $d_t(\pi)$ from π to Id_n is the minimum value k for which there exist k transpositions $\tau_1, \tau_2, \ldots, \tau_k$, satisfying:

$$\pi \circ \tau_k \circ \ldots \circ \tau_2 \circ \tau_1 = Id_n$$

The decision problem of computing the transposition distance is the following:

SORTING BY TRANSPOSITIONS PROBLEM [3] INPUT: A permutation π , an integer k. QUESTION: Is $d_t(\pi) \leq k$?

The following property directly follows from Property 3, since for any n the number of breakpoints of Id_n is 0.

Property 4. Let π be a permutation, then $d_t(\pi) \geq d_b(\pi)/3$.

Figure 2 gives an example of the computation of the transposition distance.

$$\begin{array}{lll} \pi & = & 0 \ \underline{2} \ \underline{4} \ \underline{3} \ \underline{1} \ 5 \\ \pi \circ \tau_{1,3,5} & = & 0 \ \underline{3} \ \underline{1} \ \underline{2} \ 4 \ 5 \\ \pi \circ \tau_{1,3,5} \circ \tau_{1,2,4} & = & 0 \ 1 \ 2 \ 3 \ 4 \ 5 \end{array}$$

Figure 2: The transposition distance from $\pi = (0\ 2\ 4\ 3\ 1\ 5)$ to Id_5 is 2: it is at most 2 since $\pi \circ \tau_{1,3,5} \circ \tau_{1,2,4} = Id_5$, and it cannot be less than 2 since Property 4 applies with $d_b(\pi)/3 = 5/3 > 1$.

2 3-Deletion and Transposition Operations

In this section, we introduce 3DT-instances, which are the cornerstone of our reduction from SAT to the SORTING BY TRANSPOSITIONS problem, since they are used as an intermediate between instances of the two problems. We first define 3DT-instances and the possible operations that can be applied to them, then we focus on the equivalence between these instances and permutations.

2.1 3DT-instances

Definition 4 (3DT-instance). A 3DT-instance $I = \langle \Sigma, T, \psi \rangle$ of span n is composed of the following elements:

- Σ : an alphabet;
- $T = \{(a_i, b_i, c_i) \mid 1 \leq i \leq |T|\}$: a set of (ordered) triples of elements of Σ , partitioning Σ (i.e. all elements are pairwise distinct, and $\bigcup_{i=1}^{|T|} \{a_i, b_i, c_i\} = \Sigma$);
- $\psi: \Sigma \to [1; n]$, an injection.

The domain of I is the image of ψ , that is the set $L = \{\psi(\sigma) \mid \sigma \in \Sigma\}$.

The word representation of I is the n-letter word $u_1 \ u_2 \dots u_n$ over $\Sigma \cup \{\bullet\}$ (where $\bullet \notin \Sigma$), such that for all $i \in L$, $\psi(u_i) = i$, and for $i \in [1; n] - L$, $u_i = \bullet$.

Two examples of 3DT-instances are given in Example 1. Note that such instances can be defined by their word representation and by their set of triples T. The empty 3DT-instance, in which $\Sigma = \emptyset$, can be written with a sequence of n dots, or with the empty word ε .

Example 1.

In this example, we define two 3DT-instances of span 6, $I = \langle \Sigma, T, \psi \rangle$ and $I' = \langle \Sigma', T', \psi' \rangle$:

$$I = a_1 c_2 b_1 b_2 c_1 a_2 \quad \text{with } T = \{(a_1, b_1, c_1), (a_2, b_2, c_2)\}$$

$$I' = b_2 \cdot c_2 \cdot a_2 \quad \text{with } T' = \{(a_2, b_2, c_2)\}$$

Here, *I* has an alphabet of size 6, $\Sigma = \{a_1, b_1, c_1, a_2, b_2, c_2\}$, hence ψ is a bijection $(\psi(a_1) = 1, \psi(c_2) = 2, \psi(b_1) = 3, \text{ etc})$. The second instance, *I'*, has an alphabet of size 3, $\Sigma' = \{a_2, b_2, c_2\}$, with $\psi'(b_2) = 2, \psi'(c_2) = 4, \psi'(a_2) = 6$.

Property 5. Let $I = \langle \Sigma, T, \psi \rangle$ be a 3DT-instance of span n with domain L. Then

$$|\Sigma| = |L| = 3|T| \le n.$$

Proof. We have $|\Sigma| = |L|$ since ψ is an injection with image L. The triples of T partition Σ so $|\Sigma| = 3|T|$, and finally $L \subseteq [1; n]$ so $|L| \le n$.

Definition 5. Let $I = \langle \Sigma, T, \psi \rangle$ be a 3DT-instance. The injection ψ gives a total order over Σ , written \prec_I (or \prec , if there is no ambiguity), defined by

$$\forall \sigma_1, \sigma_2 \in \Sigma, \quad \sigma_1 \prec_I \sigma_2 \Leftrightarrow \psi(\sigma_1) < \psi(\sigma_2) \tag{1}$$

Two elements σ_1 and σ_2 of Σ are called consecutive if there exists no element $x \in \Sigma$ such that $\sigma_1 \prec_I x \prec_I \sigma_2$. In this case, we write $\sigma_1 \triangleleft_I \sigma_2$ (or simply $\sigma_1 \triangleleft \sigma_2$).

An equivalent definition is that $\sigma_1 \prec \sigma_2$ if σ_1 σ_2 is a subword of the word representation of I. Also, $\sigma_1 \triangleleft \sigma_2$ if the word representation of I contains a factor of the kind $\sigma_1 \cdot \sigma_2$ (where $\sigma_1 \cdot \sigma_2$ (where $\sigma_1 \cdot \sigma_2$ (where $\sigma_2 \cdot \sigma_3$ is a subword of the word representation of $\sigma_3 \cdot \sigma_4$ is a subword of the word representation of $\sigma_4 \cdot \sigma_4$ is a subword of the word representation of $\sigma_4 \cdot \sigma_4$ is a subword of the word representation of $\sigma_4 \cdot \sigma_4$ is a subword of the word representation of $\sigma_4 \cdot \sigma_4$ is a subword of the word representation of $\sigma_4 \cdot \sigma_4$ is a subword of the word representation of $\sigma_4 \cdot \sigma_4$ is a subword of the word representation of $\sigma_4 \cdot \sigma_4$ is a subword of the word representation of $\sigma_4 \cdot \sigma_4$ is a subword of the word representation of $\sigma_4 \cdot \sigma_4$ is a subword of the word representation of $\sigma_4 \cdot \sigma_4$ is a subword of the word representation of $\sigma_4 \cdot \sigma_4$ is a subword of $\sigma_4 \cdot \sigma_4$ in $\sigma_4 \cdot \sigma_4$ is a subword of $\sigma_4 \cdot \sigma_4$ is a subword of $\sigma_4 \cdot \sigma_4$ in $\sigma_4 \cdot \sigma_4$ is a subword of $\sigma_4 \cdot \sigma_4$ in $\sigma_4 \cdot \sigma_4$ is a subword of $\sigma_4 \cdot \sigma_4$ in $\sigma_4 \cdot \sigma_4$ in $\sigma_4 \cdot \sigma_4$ in $\sigma_4 \cdot \sigma_4$ is a subword of $\sigma_4 \cdot \sigma_4$ in $\sigma_$

Using the triples in T, we define a successor function over the domain L:

Definition 6. Let $I = \langle \Sigma, T, \psi \rangle$ be a 3DT-instance with domain L. The function $succ_I : L \to L$ is defined by:

$$\forall (a, b, c) \in T, \quad \psi(a) \mapsto \psi(b)$$
$$\psi(b) \mapsto \psi(c)$$
$$\psi(c) \mapsto \psi(a)$$

Function $succ_I$ is a bijection, with no fixed-points, and such that $succ_I \circ succ_I \circ succ_I$ is the identity over L. In Example 1, we have:

$$succ_I = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 5 & 2 & 1 & 4 \end{pmatrix}$$
 and $succ_{I'} = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{pmatrix}$.

2.2 3DT-steps

Definition 7. Let $I = \langle \Sigma, T, \psi \rangle$ be a 3DT-instance, and (a, b, c) be a triple of T. Write $i = \min\{\psi(a), \psi(b), \psi(c)\}$, $j = succ_I(i)$, and $k = succ_I(j)$. The triple $(a, b, c) \in T$ is well-ordered if we have i < j < k. In such a case, we write $\tau[a, b, c, \psi]$ the transposition $\tau_{i,j,k}$.

An equivalent definition is that (a,b,c) is well-ordered iff one of abc, bca, cab is a subword of the word representation of I. In Example 1, (a_1,b_1,c_1) is well-ordered in I: indeed, we have $i=\psi(a_1),\ j=\psi(b_1)$ and $k=\psi(c_1),\ so\ i< j< k$. The triple (a_2,b_2,c_2) is also well-ordered in I' $(i=\psi'(b_2)< j=\psi'(c_2)< k=\psi'(a_2))$, but not in I: $i=\psi(c_2)< k=\psi(b_2)< j=\psi(a_2)$. In this example, we have $\tau[a_1,b_1,c_1,\psi]=\tau_{1,3,5}$ and $\tau[a_2,b_2,c_2,\psi']=\tau_{2,4,6}$.

Definition 8 (3DT-step). Let $I = \langle \Sigma, T, \psi \rangle$ be a 3DT-instance with $(a, b, c) \in T$ a well-ordered triple. The 3DT-step of parameter (a, b, c) is the operation written $\xrightarrow{(a, b, c)}$, transforming I into the 3DT-instance $I' = \langle \Sigma', T', \psi' \rangle$ such that:

- $\Sigma' = \Sigma \{a, b, c\}$
- $T' = T \{(a, b, c)\}$

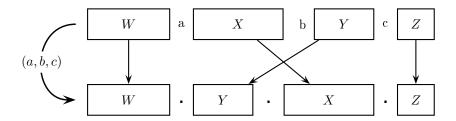


Figure 3: The 3DT-step $\xrightarrow{(a,b,c)}$ has two effects, here represented on the word representation of a 3DT-instance: the triple (a,b,c) is deleted (and replaced by dots in this word representation), and the factors X and Y are swapped.

•
$$\psi'$$
: $\begin{array}{ccc} \Sigma' & \to & \llbracket 1 \, ; \, n \rrbracket \\ \sigma & \mapsto & \tau^{-1}(\psi(\sigma)) \end{array}$ $(with \ \tau = \tau[a,b,c,\psi]).$

A 3DT-step has two effects on a 3DT-instance, as represented in Figure 3. The first is to remove a necessarily well-ordered triple from T (hence from Σ). The second is, by applying a transposition to ψ , to shift the position of some of the remaining elements. Note that a triple that is not well-ordered in I can become well-ordered in I', or vice-versa. In Example 1, I' can be obtained from I via a 3DT-step: $I^{(a_1,b_1,c_1)}I'$. Moreover, $I'^{(a_2,b_2,c_2)}\varepsilon$. A more complex example is given in Figure 4.

Note that a 3DT-step transforms the function $succ_I$ into $succ_{I'} = \tau^{-1} \circ succ_I \circ \tau$, restricted to L', the domain of the new instance I'. Indeed, for all $(a, b, c) \in T'$, we have

$$succ_{I'}(\psi'(a)) = \psi'(b)$$

$$= \tau^{-1}(\psi(b))$$

$$= \tau^{-1}(succ_I(\psi(a)))$$

$$= \tau^{-1}(succ_I(\tau(\psi'(a))))$$

$$= (\tau^{-1} \circ succ_I \circ \tau)(\psi'(a))$$

The computation is similar for $\psi'(b)$ and $\psi'(c)$.

Definition 9 (3DT-collapsibility). A 3DT-instance $I = \langle \Sigma, T, \psi \rangle$ is 3DT-collapsible if there exists a sequence of 3DT-instances $I_k, I_{k-1}, \ldots, I_0$ such that

$$I_k = I$$

$$\forall i \in [1; k], \quad \exists (a, b, c) \in T, \quad I_i \xrightarrow{(a, b, c)} I_{i-1}$$

$$I_0 = \varepsilon$$

In Example 1, I and I' are 3DT-collapsible, since $I^{(a_1,b_1,c_1)}I'^{(a_2,b_2,c_2)}\varepsilon$. Another example is the 3DT-instance defined in Figure 4. Note that in the example of Figure 4, there are in fact two distinct paths leading to the empty instance.

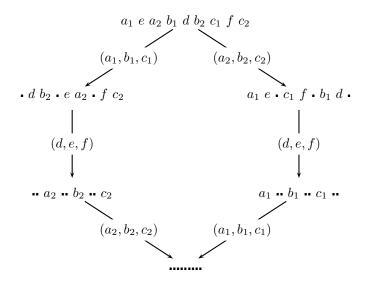


Figure 4: Possible 3DT-steps from the instance I defined by the word a_1 e a_2 b_1 d b_2 c_1 f c_2 and the set of triples $T = \{(a_1, b_1, c_1), (a_2, b_2, c_2), (d, e, f)\}$. We can see that there is a path from I to ε , hence I is 3DT-collapsible. Note that both (a_1, b_1, c_1) and (a_2, b_2, c_2) are well-ordered in the initial instance, each one loses this property after applying the 3DT-step associated to the other, and becomes well-ordered again after applying the 3DT-step associated to (d, e, f).

2.3 Equivalence with the transposition distance

Definition 10. Let $I = \langle \Sigma, T, \psi \rangle$ be a 3DT-instance of span n with domain L, and π be a permutation of [0, n]. We say that I and π are equivalent, and we write $I \sim \pi$, if:

$$\begin{aligned} &\pi(0) = 0,\\ \forall v \in \llbracket 1\,;\, n \rrbracket - L, &\pi(v) = \pi(v-1) + 1,\\ \forall v \in L, &\pi(v) = \pi(succ_I^{-1}(v) - 1) + 1. \end{aligned}$$

With such an equivalence $I \sim \pi$, the two following properties hold:

- The breakpoints of π correspond to the elements of L (see Property 6).
- The triples of breakpoints that may be resolved immediately by a single transposition correspond to the well-ordered triples of T (see Figure 5 and Lemma 8).

Property 6. Let $I = \langle \Sigma, T, \psi \rangle$ be a 3DT-instance of span n with domain L, and π be a permutation of [0, n], such that $I \sim \pi$. Then the number of breakpoints of π is $d_b(\pi) = |L| = 3|T|$.

Proof. Let $v \in [1; n]$. By Definition 10, we have:

If $v \notin L$, then $\pi(v) = \pi(v-1) + 1$, so (v-1,v) is an adjacency of π .

If $v \in L$, we write $u = succ_I^{-1}(v)$, so $\pi(v) = \pi(u-1) + 1$. Since $succ_I$ has no fixed-point, we have $u \neq v$, which implies $\pi(u-1) \neq \pi(v-1)$. Hence, $\pi(v) \neq \pi(v-1) + 1$, and (v-1,v) is a breakpoint of π .

Consequently the number of breakpoints of π is exactly |L|, and |L| = 3|T| by Property 5. \square

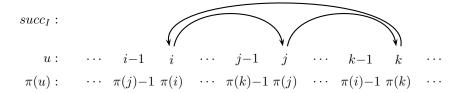


Figure 5: Illustration of the equivalence $I \sim \pi$ on three integers (i, j, k) such that $j = succ_I(i)$ and $k = succ_I(j)$. It can be checked that $\pi(v) = \pi(u-1) + 1$ for any $(u, v) \in \{(i, j), (j, k), (k, i)\}$.

Figure 6: Illustration of Lemma 7: since $I \sim \pi$ and $I \xrightarrow{(a_1,b_1,c_1)} I'$, then $I' \sim \pi' = \pi \circ \tau$, where $\tau = \tau[a_1,b_1,c_1,\psi]$.

With the following lemma, we show that the equivalence between a 3DT-instance and a permutation is preserved after a 3DT-step, see Figure 6.

Lemma 7. Let $I = \langle \Sigma, T, \psi \rangle$ be a 3DT-instance of span n, and π be a permutation of [0; n], such that $I \sim \pi$. If there exists a 3DT-step $I \xrightarrow{(a,b,c)} I'$, then I' and $\pi' = \pi \circ \tau$, where $\tau = \tau[a,b,c,\psi]$, are equivalent.

Proof. We write (i, j, k) the indices such that $\tau = \tau_{i,j,k}$ (i.e. $i = \min\{\psi(a), \psi(b), \psi(c)\}$, $j = succ_I(i)$, $k = succ_I(j)$). Since (a, b, c) is well-ordered, we have i < j < k.

We have $I' = \langle \Sigma', T', \psi' \rangle$, with $\Sigma' = \Sigma - \{a, b, c\}$, $T' = T - \{(a, b, c)\}$, and $\psi' : \sigma \mapsto \tau^{-1}(\psi(\sigma))$. We write respectively L and L' the domains of I and I'. For all $v' \in [1; n]$, we have

$$v' \in L' \Leftrightarrow \exists \sigma \in \Sigma - \{a, b, c\}, \ v' = \tau^{-1}(\psi(\sigma))$$

 $\Leftrightarrow \tau(v') \in L - \{i, j, k\}$

We prove the 3 required properties (see Definition 10) sequentially:

- $\pi'(0) = \pi(\tau(0)) = \pi(0) = 0$,
- $\forall v' \in [1; n] L'$, let $v = \tau(v')$. Since $v' \notin L'$, we have either $v \in \{i, j, k\}$, or $v \notin L$. In the first case, we write $u = succ_I^{-1}(v)$ (then $u \in \{i, j, k\}$). By Property 2, $\tau^{-1}(u-1)$ is equal to

$$\tau^{-1}(succ_I(u)) - 1$$
, so $\tau^{-1}(u - 1) = \tau^{-1}(v) - 1$. Hence,
 $\pi'(v' - 1) + 1 = \pi(\tau(\tau^{-1}(v) - 1)) + 1$
 $= \pi(u - 1) + 1$
 $= \pi(v)$ by Def. 10, since $v \in L$ and $v = succ_I(u)$
 $= \pi'(v')$

In the second case, $v \notin L$, we have

$$\pi'(v'-1) + 1 = \pi(\tau(\tau^{-1}(v)-1)) + 1$$

$$= \pi(\tau(\tau^{-1}(v-1))) + 1 \text{ by Prop. 2, since } v \notin \{i, j, k\}$$

$$= \pi(v-1) + 1$$

$$= \pi(v) \text{ by Def. 10, since } v \notin L$$

$$= \pi'(v')$$

In both cases, we indeed have $\pi'(v'-1)+1=\pi'(v')$.

• Let v' be an element of L'. We write $v = \tau(v')$, $u = succ_I^{-1}(v)$, and $u' = \tau^{-1}(u)$. Then $v' = \tau^{-1}(succ_I(\tau(u'))) = succ_{I'}(u')$. Moreover, $v \notin \{i, j, k\}$, hence $u \notin \{i, j, k\}$.

$$\pi'(u'-1) + 1 = \pi(\tau(\tau^{-1}(u)-1)) + 1$$

$$= \pi(\tau(\tau^{-1}(u-1))) + 1 \text{ by Prop. 2, since } u \notin \{i,j,k\}$$

$$= \pi(u-1) + 1$$

$$= \pi(v) \text{ by Def. 10, since } v \in L \text{ and } u = succ_I^{-1}(v)$$

$$= \pi(\tau(\tau^{-1}(v)))$$

$$= \pi'(v')$$

Lemma 8. Let $I = \langle \Sigma, T, \psi \rangle$ be a 3DT-instance of span n, and π a permutation of [0, n], such that $I \sim \pi$. If there exists a transposition $\tau = \tau_{i,j,k}$ such that $d_b(\pi \circ \tau) = d_b(\pi) - 3$, then T contains a well-ordered triple (a, b, c) such that $\tau = \tau[a, b, c, \psi]$.

Proof. We write $i' = \tau^{-1}(i)$, $j' = \tau^{-1}(j)$, and $k' = \tau^{-1}(k)$. Note that i < j < k.

Let $\pi' = \pi \circ \tau$. For all $x \in [1; n] - \{i, j, k\}$, we have, by Property 3, that (x - 1, x) is an adjacency of π iff $(\tau^{-1}(x) - 1, \tau^{-1}(x))$ is an adjacency of π' . Hence, since $d_b(\pi') = d_b(\pi) - 3$, we necessarily have that (i - 1, i), (j - 1, j) and (k - 1, k) are breakpoints of π , and (i' - 1, i'), (j' - 1, j') and (k' - 1, k') are adjacencies of π' . We have

$$\pi(i) = \pi(\tau(i'))$$

$$= \pi'(i')$$

$$= \pi'(i'-1) + 1 \text{ since } (i'-1,i') \text{ is an adjacency of } \pi'$$

$$= \pi'(\tau^{-1}(i)-1) + 1$$

$$= \pi'(\tau^{-1}(k-1)) + 1 \text{ by Prop. 2}$$

$$= \pi(k-1) + 1$$

Since $I \sim \pi$ and $i \neq k$, by Definition 10, we necessarily have $i \in L$ (where L is the domain of I), and $i = succ_I(k)$.

Using the same method with (j'-1,j') and (k'-1,k'), we obtain $j,k \in L$, $j = succ_I(i)$ and $k = succ_I(j)$. Hence, T contains one of the following three triples: $(\psi^{-1}(i), \psi^{-1}(j), \psi^{-1}(k))$, $(\psi^{-1}(j), \psi^{-1}(k), \psi^{-1}(i))$ or $(\psi^{-1}(k), \psi^{-1}(i), \psi^{-1}(j))$. Writing (a, b, c) this triple, we indeed have $\tau_{i,j,k} = \tau[a,b,c,\psi]$ since i < j < k.

Theorem 9. Let $I = \langle \Sigma, T, \psi \rangle$ be a 3DT-instance of span n with domain L, and π be a permutation of [0, n], such that $I \sim \pi$. Then I is 3DT-collapsible if and only if $d_t(\pi) = |T| = d_b(\pi)/3$.

Proof. We prove the theorem by induction on k = |T|. For k = 0, necessarily $I = \varepsilon$ and $L = T = \emptyset$, and by Definition 10, $\pi = Id_n$ ($\pi(0) = 0$, and for all v > 0, $\pi(v) = \pi(v - 1) + 1$). In this case, I is trivially 3DT-collapsible, and $d_t(\pi) = 0 = |T| = d_b(\pi)/3$.

Suppose now k = k' + 1, with $k' \ge 0$, and the theorem is true for k'. By Property 6, we have $d_b(\pi) = 3k$, and by Property 4, $d_t(\pi) \ge 3k/3 = k$.

Assume first that I is 3DT-collapsible. Then there exist both a triple $(a,b,c) \in T$ and a 3DT-instance $I' = \langle \Sigma', T', \psi' \rangle$ such that $I \xrightarrow{(a,b,c)} I'$, and that I' is 3DT-collapsible. Since $T' = T - \{(a,b,c)\}$, the size of T' is k-1 = k'. By Lemma 7, we have $I' \sim \pi' = \pi \circ \tau$, with $\tau = \tau[a,b,c,\psi]$. Using the induction hypothesis, we know that $d_t(\pi') = k'$. So the transposition distance from $\pi = \pi' \circ \tau^{-1}$ to the identity is at most, hence exactly, k' + 1 = k.

Assume now that $d_t(\pi) = k$. We can decompose π into $\pi = \pi' \circ \tau^{-1}$, where τ is a transposition and π' a permutation such that $d_t(\pi') = k - 1 = k'$. Since π has 3k breakpoints (Property 6), and $\pi' = \pi \circ \tau$ has at most 3k - 3 breakpoints (Property 4), τ necessarily removes 3 breakpoints, and we can use Lemma 8: there exists a 3DT-step $I^{\underline{(a,b,c)}}I'$, where $(a,b,c) \in T$ is a well-ordered triple and $\tau = \tau[a,b,c,\psi]$. We can now use Lemma 7, which yields $I' \sim \pi' = \pi \circ \tau$. Using the induction hypothesis, we obtain that I' is 3DT-collapsible, hence I is also 3DT-collapsible. This concludes the proof of the theorem.

The previous theorem gives a way to reduce the problem of deciding if a 3DT-instance is collapsible to the SORTING BY TRANSPOSITIONS problem. However, it must be used carefully, since there exist 3DT-instances to which no permutation is equivalent (for example, $I = a_1 \ a_2 \ b_1 \ b_2 \ c_1 \ c_2$ admits no permutation π of [0; 6] such that $I \sim \pi$).

3 3DT-collapsibility is NP-Hard to Decide

In this section, we define, for any boolean formula ϕ , a corresponding 3DT-instance I_{ϕ} . We also prove that I_{ϕ} is 3DT-collapsible if and only if ϕ is satisfiable.

3.1 Block Structure

The construction of the 3DT-instance I_{ϕ} uses a decomposition into blocks, defined below. Some triples will be included in a block, in order to define its behavior, while others will be shared between two blocks, in order to pass information. The former are unconstrained, so that we can design blocks with the behavior we need (for example, blocks mimicking usual boolean functions), while the latter need to follow several rules, so that the blocks can conveniently be arranged together.

Definition 11 (*l*-block-decomposition). An *l*-block-decomposition \mathcal{B} of a 3DT-instance I of span n is an l-tuple (s_1, \ldots, s_l) such that $s_1 = 0$, for all $h \in [1; l-1]$, $s_h < s_{h+1}$ and $s_l < n$. We write $t_h = s_{h+1}$ for $h \in [1; l-1]$, and $t_l = n$.

Let $I = \langle \Sigma, T, \psi \rangle$. For $h \in [1; l]$, the factor $u_{s_h+1} \ u_{s_h+2} \ \dots u_{t_h}$ of the word representation $u_1 \ u_2 \ \dots u_n$ of I is called the full block \mathcal{B}_h^{\bullet} (it is a word over $\Sigma \cup \{\bullet\}$). The subword of \mathcal{B}_h^{\bullet} where every occurrence of \bullet is deleted is called the block \mathcal{B}_h .

For $\sigma \in \Sigma$, we write $block_{I,\mathcal{B}}(\sigma) = h$ if $\psi(\sigma) \in [s_h + 1; t_h]$ (equivalently, if σ appears in the word \mathcal{B}_h). A triple $(a,b,c) \in T$ is said to be internal if $block_{I,\mathcal{B}}(a) = block_{I,\mathcal{B}}(b) = block_{I,\mathcal{B}}(c)$, external otherwise.

If τ is a transposition such that for all $h \in [1; l]$, $\tau(s_h) < \tau(t_h)$, we write $\tau[\mathcal{B}]$ the l-block-decomposition $(\tau(s_1), \ldots, \tau(s_l))$.

In the rest of this section, we mostly work with blocks instead of full blocks, since we are only interested in the relative order of the elements, rather than their actual position. Full blocks are only used in definitions, where we want to control the dots in the word representation of the 3DT-instances we define. Note that, for $\sigma_1, \sigma_2 \in \Sigma$ such that $block_{I,\mathcal{B}}(\sigma_1) = block_{I,\mathcal{B}}(\sigma_2) = h$, the relation $\sigma_1 \triangleleft \sigma_2$ is equivalent to $\sigma_1 \sigma_2$ is a factor of \mathcal{B}_h .

Property 10. Let $\mathcal{B} = (s_1, \ldots, s_l)$ be an l-block-decomposition of a 3DT-instance of span n, and $i, j, k \in [1; n]$ be three integers such that (a) i < j < k and (b) $\exists h_0$ such that $s_{h_0} < i < j \le t_{h_0}$ or $s_{h_0} < j < k \le t_{h_0}$ (or both). Then for all $h \in [1; l]$, $\tau_{i,j,k}^{-1}(s_h) < \tau_{i,j,k}^{-1}(t_h)$, and the l-block-decomposition $\tau_{i,j,k}^{-1}[\mathcal{B}]$ is defined.

Proof. For any $h \in [1; l]$, we show that we cannot have $i \leq s_h < j \leq t_h < k$. Indeed, $s_h < j$ implies $h \leq h_0$ (since $s_h < j \leq t_{h_0} = s_{h_0+1}$), and $j \leq t_h$ implies $h \geq h_0$ (since $t_{h_0-1} = s_{h_0} < j \leq t_h$). Hence $s_h < j \leq t_h$ implies $h = h_0$, but $i \leq s_h, t_h < k$ contradicts both conditions $s_{h_0} < i$ and $k \leq t_{h_0}$: hence the relation $i \leq s_h < j \leq t_h < k$ is impossible.

By Property 1, since $s_h < t_h$ for all $h \in [1; l]$, and $i \le s_h < j \le t_h < k$ does not hold, we have $\tau_{i,j,k}^{-1}(s_h) < \tau_{i,j,k}^{-1}(t_h)$, which is sufficient to define $\tau_{i,j,k}^{-1}[\mathcal{B}]$.

The above property yields that, if (a, b, c) is a well-ordered triple of a 3DT-instance $I = \langle \Sigma, T, \psi \rangle$ $(\tau = \tau[a, b, c, \psi])$, and \mathcal{B} is an l-block-decomposition of I, then $\tau^{-1}[\mathcal{B}]$ is defined if (a, b, c) is an internal triple, or an external triple such that one of the following equalities is satisfied: $block_{I,\mathcal{B}}(a) = block_{I,\mathcal{B}}(b)$, $block_{I,\mathcal{B}}(b) = block_{I,\mathcal{B}}(c)$ or $block_{I,\mathcal{B}}(c) = block_{I,\mathcal{B}}(a)$. In this case, the

3DT-step $I \xrightarrow{(a,b,c)} I'$ is written $(I,\mathcal{B}) \xrightarrow{(a,b,c)} (I',\mathcal{B}')$, where $\mathcal{B}' = \tau^{-1}[\mathcal{B}]$ is an l-block-decomposition of I'.

Definition 12 (Variable). A variable A of a 3DT-instance $I = \langle \Sigma, T, \psi \rangle$ is a pair of triples A = [(a, b, c), (x, y, z)] of T. It is valid in an l-block-decomposition \mathcal{B} if

- (i) $\exists h_0 \in \llbracket 1; l \rrbracket$ such that $block_{I,\mathcal{B}}(b) = block_{I,\mathcal{B}}(x) = block_{I,\mathcal{B}}(y) = h_0$
- (ii) $\exists h_1 \in [1; l], h_1 \neq h_0$, such that $block_{I,\mathcal{B}}(a) = block_{I,\mathcal{B}}(c) = block_{I,\mathcal{B}}(z) = h_1$
- (iii) if $x \prec y$, then we have $x \triangleleft b \triangleleft y$
- (iv) $a \prec z \prec c$

For such a valid variable A, the block \mathcal{B}_{h_0} containing $\{b, x, y\}$ is called the source of A (we write $source(A) = h_0$), and the block \mathcal{B}_{h_1} containing $\{a, c, z\}$ is called the target of A (we write $target(A) = h_1$). For $h \in [1; l]$, the variables of which \mathcal{B}_h is the source (resp. the target) are called the output (resp. the input) of \mathcal{B}_h . The 3DT-step $I \xrightarrow{(x,y,z)} I'$ is called the activation of A (it requires that (x,y,z) is well-ordered).

Note that since a valid variable A = [(a, b, c), (x, y, z)] has $block_{I,\mathcal{B}}(x) = block_{I,\mathcal{B}}(y)$, its activation can be written $(I, \mathcal{B}) \xrightarrow{(x, y, z)} (I', \mathcal{B}')$.

Property 11. Let (I, \mathcal{B}) be a 3DT-instance with an l-block-decomposition, and A be a variable of I that is valid in \mathcal{B} . Write A = [(a, b, c), (x, y, z)]. Then (x, y, z) is well-ordered iff $x \prec y$; and (a, b, c) is not well-ordered.

Proof. Note that for all $\sigma, \sigma' \in \Sigma$, $block_{I,\mathcal{B}}(\sigma) < block_{I,\mathcal{B}}(\sigma') \Rightarrow \sigma \prec \sigma'$. Write $I = \langle \Sigma, T, \psi \rangle$, $h_0 = source(A)$ and $h_1 = target(A)$: we have $h_0 \neq h_1$ by condition (ii) of Definition 12.

If $h_0 < h_1$, then, with condition (iv) of Definition 12, $b \prec a \prec c$, and either $x \prec y \prec z$ or $y \prec x \prec z$. Hence, (a, b, c) is not well-ordered, and (x, y, z) is well-ordered iff $x \prec y$.

Likewise, if $h_1 < h_0$, we have $a \prec c \prec b$, and $z \prec x \prec y$ or $z \prec y \prec x$. Again, (a, b, c) is not well-ordered, and (x, y, z) is well-ordered iff $x \prec y$.

Property 12. Let (I, \mathcal{B}) be a 3DT-instance with an l-block-decomposition, such that the external triples of $I = \langle \Sigma, T, \psi \rangle$ can be partitioned into a set of valid variables \mathcal{A} . Let (d, e, f) be a well-ordered triple of I, such that there exists a 3DT-step $(I, \mathcal{B}) \xrightarrow{(d, e, f)} (I', \mathcal{B}')$, with $I' = \langle \Sigma', T', \psi' \rangle$. Then one of the two following cases is true:

- (d, e, f) is an internal triple. We write $h_0 = block_{I,\mathcal{B}}(d) = block_{I,\mathcal{B}}(e) = block_{I,\mathcal{B}}(f)$. Then for all $\sigma \in \Sigma'$, $block_{I',\mathcal{B}'}(\sigma) = block_{I,\mathcal{B}}(\sigma)$. Moreover if $\sigma_1, \sigma_2 \in \Sigma'$ with $block_{I',\mathcal{B}'}(\sigma_1) = block_{I',\mathcal{B}'}(\sigma_2) \neq h_0$ and $\sigma_1 \prec_I \sigma_2$, then $\sigma_1 \prec_{I'} \sigma_2$.
- $\exists A = [(a,b,c),(x,y,z)] \in \mathcal{A}$, such that (d,e,f) = (x,y,z). Then $block_{I',\mathcal{B}'}(b) = target(A)$ and for all $\sigma \in \Sigma' \{b\}$, $block_{I',\mathcal{B}'}(\sigma) = block_{I,\mathcal{B}}(\sigma)$. Moreover if $\sigma_1, \sigma_2 \in \Sigma' \{b\}$, such that $\sigma_1 \prec_I \sigma_2$, then $\sigma_1 \prec_{I'} \sigma_2$.

Proof. We respectively write τ and i, j, k the transposition and the three integers such that $\tau = \tau_{i,j,k} = \tau[d, e, f, \psi]$ (necessarily, $0 < i < j < k \le n$). We also write $\mathcal{B} = (s_0, s_1, \ldots, s_l)$. The triple (d, e, f) is either internal or external in \mathcal{B} .

If (d, e, f) is internal, with $h_0 = block_{I,\mathcal{B}}(d) = block_{I,\mathcal{B}}(e) = block_{I,\mathcal{B}}(f)$, we have (see Figure 7a):

$$s_{h_0} < i < j < k \le t_{h_0}$$
.

Hence for all $h \in [1; l]$, either $s_h < i$ or $k \le s_h$, and $\tau^{-1}(s_h) = s_h$ by Definition 1. Moreover, for all $\sigma \in \Sigma$, we have

$$i \leq \psi(\sigma) < k \quad \Rightarrow \quad \psi(\sigma) \in \llbracket s_{h_0} + 1 \, ; \, t_{h_0} \rrbracket \text{ and } \tau^{-1}(s_{h_0}) < i \leq \tau^{-1}(\psi(\sigma)) < k \leq \tau^{-1}(t_{h_0})$$

$$\Rightarrow \quad block_{I,\mathcal{B}}(\sigma) = h_0 = block_{I',\mathcal{B}'}(\sigma)$$

$$\psi(\sigma) < i \text{ or } k \leq \psi(\sigma) \quad \Rightarrow \quad \tau^{-1}(\psi(\sigma)) = \psi(\sigma)$$

$$\Rightarrow \quad block_{I',\mathcal{B}'}(\sigma) = block_{I,\mathcal{B}}(\sigma)$$

Finally, if $\sigma_1, \sigma_2 \in \Sigma'$ with $block_{I',\mathcal{B}'}(\sigma_1) = block_{I',\mathcal{B}'}(\sigma_2) \neq h_0$, then we have both $\tau^{-1}(\psi(\sigma_1)) = \psi(\sigma_1)$ and $\tau^{-1}(\psi(\sigma_2)) = \psi(\sigma_2)$. Hence $\sigma_1 \prec_I \sigma_2 \Leftrightarrow \sigma_1 \prec_{I'} \sigma_2$.

If (d, e, f) is external, then, since the set of external triples can be partitioned into variables, there exists a variable $A = [(a, b, c), (x, y, z)] \in \mathcal{A}$, such that (d, e, f) = (a, b, c) or (d, e, f) = (x, y, z). Since (d, e, f) is well-ordered in I, we have, by Property 11, (d, e, f) = (x, y, z) and $x \prec_I y$, see Figure 7b. And since A is valid, by condition (iv) of Definition 12, $x \triangleleft_I b \triangleleft_I y$. We write $h_0 = source(A)$ and $h_1 = target(A)$, and we assume that $h_0 < h_1$, which implies $x \prec_I y \prec_I z$ (the case $h_1 < h_0$ with $z \prec_I x \prec_I y$ is similar): thus, we have

$$i = \psi(x), \ j = \psi(y), \ k = \psi(z), \ \text{and} \ s_{h_0} < i < j \le t_{h_0} \le s_{h_1} < k \le t_{h_1}.$$

We define a set of indices U by

$$U = \{s_h \mid h \in [1; l]\} \cup \{n\} \cup \{\psi(\sigma) \mid \sigma \in \Sigma' - \{b\}\}.$$

We now show that for all $u \in U$, we have u < i or $j \le u$. Indeed, if $u = s_h$ for some $h \in [1; l]$, then either $h \le h_0$ and $u \le s_{h_0} < i$, or $h_0 < h$ and $j \le t_{h_0} \le u$. Also, if u = n, then $j \le u$. Finally, assume $u = \psi(\sigma)$, with $\sigma \in \Sigma' - \{b\}$. We then have $x \prec_I \sigma \prec_I y \Leftrightarrow \sigma = b$, since $x \triangleleft_I b \triangleleft_I y$. Hence either $\sigma \prec_I x$ and $u < \psi(x) = i$, or $y \prec_I \sigma$ and $\psi(y) = j < u$.

By Property 1, if $u, v \in U$ are such that u < v, then $\tau^{-1}(u) < \tau^{-1}(v)$. This implies that elements of $\Sigma' - \{b\} = \Sigma - \{b, x, y, z\}$ do not change blocks after applying τ^{-1} on ψ , and that the relative order of any two elements is preserved. Finally, for b, we have $x \prec_I b \prec_I y$, hence

$$i \le \psi(b) < j \le s_{h_1} < k \le t_{h_1}.$$

Thus, by Property 1, $\tau^{-1}(s_{h_1}) < \tau^{-1}(\psi(b)) < \tau^{-1}(t_{h_1})$, and $block_{I',\mathcal{B}'}(b) = h_1 = target(A)$. This completes the proof.

Definition 13 (Valid context). A 3DT-instance with an l-block-decomposition (I, \mathcal{B}) is a valid context if the set of external triples of I can be partitioned into valid variables.

With the following property, we ensure that a valid context remains *almost* valid after applying a 3DT-step: the partition of the external triples into variables if kept through this 3DT-step, but conditions (iii) and (iv) of Definition 12 are not necessarily satisfied.

Property 13. Let (I, \mathcal{B}) be a valid context and $(I, \mathcal{B}) \xrightarrow{(d, e, f)} (I', \mathcal{B}')$ be a 3DT-step. Then the external triples of (I', \mathcal{B}') can be partitioned into a set of variables, each satisfying conditions (i) and (ii) of Definition 12.

Proof. Let $I = \langle \Sigma, T, \psi \rangle$, $I' = \langle \Sigma', T', \psi' \rangle$, \mathcal{A} be the set of variables of I, and E (resp. E') be the set of external triples of I (resp. I'). From Property 12, two cases are possible.

First case: $(d, e, f) \notin E$. Then for all $\sigma \in \Sigma'$, $block_{I',\mathcal{B}'}(\sigma) = block_{I,\mathcal{B}}(\sigma)$. Hence E' = E, and (I', \mathcal{B}') has the same set of variables as (I, \mathcal{B}) , that is \mathcal{A} . The source and target blocks of every variable remain unchanged, hence conditions (i) and (ii) of Definition 12 are still satisfied for each $A \in \mathcal{A}$ in \mathcal{B}' .

Second case: $(d, e, f) \in E$, and $\exists A = [(a, b, c), (x, y, z)] \in \mathcal{A}$, such that (d, e, f) = (x, y, z), by Property 12. Then $block_{I',\mathcal{B}'}(b) = target(A)$ and for all $\sigma \in \Sigma' - \{b\}$, $block_{I',\mathcal{B}'}(\sigma) = block_{I,\mathcal{B}}(\sigma)$. Hence $block_{I',\mathcal{B}'}(b) = block_{I',\mathcal{B}'}(a) = block_{I',\mathcal{B}'}(c)$, and $E' = E - \{(x, y, z), (a, b, c)\}$: indeed, (x, y, z) is deleted in T' so $(x, y, z) \notin E'$, (a, b, c) is internal in I', and every other triple is untouched. And for every $A' = [(a', b', c'), (x', d', e')] \in \mathcal{A} - \{A\}$, we have $block_{I',\mathcal{B}'}(\sigma) = block_{I,\mathcal{B}}(\sigma)$ for $\sigma \in \{a', b', c', x', y', z'\}$, hence A' satisfies conditions (i) and (ii) of Definition 12 in \mathcal{B}' .

Consider a block B in a valid context (I, \mathcal{B}) (there exists $h \in [1; l]$ such that $B = \mathcal{B}_h$), and (d, e, f) a triple of I such that $(I, \mathcal{B}) \xrightarrow{(d, e, f)} (I', \mathcal{B}')$ (we write $B' = \mathcal{B}'_h$). Then, following Property 12, four cases are possible:

- $h \notin \{block_{I,\mathcal{B}}(d), block_{I,\mathcal{B}}(e), block_{I,\mathcal{B}}(f)\}$, hence B' = B, since, by Property 12, the relative order of the elements of B remains unchanged after the 3DT-step $\xrightarrow{(d,e,f)}$.
- (d, e, f) is an internal triple of B. We write

$$B \longrightarrow (d, e, f) \longrightarrow B'$$

• $\exists A = [(a,b,c),(x,y,z)]$ such that h = source(A) and (d,e,f) = (x,y,z) (A is an output of B), see Figure 8 (left). We write

$$B \longrightarrow A \longrightarrow B'$$

• $\exists A = [(a, b, c), (x, y, z)]$ such that h = target(A) and (d, e, f) = (x, y, z) (A is an input of B), see Figure 8 (right). We write

$$B \longrightarrow A \longrightarrow B'$$

The graph obtained from a block B by following exhaustively the possible arcs as defined above (always assuming this block is in a valid context) is called the *behavior graph* of B.

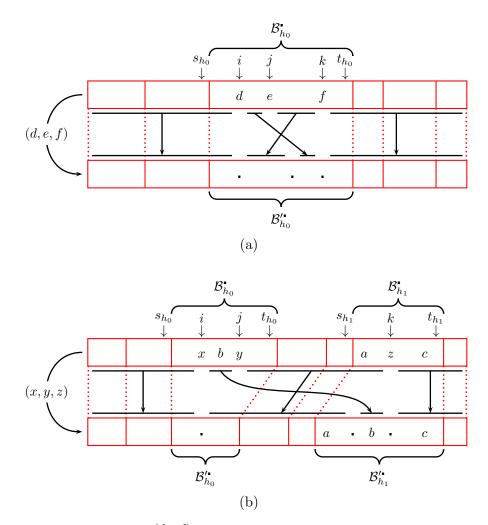


Figure 7: Effects of a 3DT-step $\xrightarrow{(d,e,f)}$ on an l-block-decomposition if (a) (d,e,f) is an internal triple, or (b) there exists a variable A=[(a,b,c),(x,y,z)] such that (d,e,f)=(x,y,z). Both figures are in fact derived from Figure 3 in the context of an l-block-decomposition.

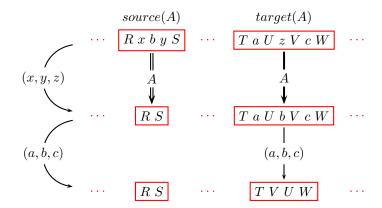


Figure 8: The activation of a variable A = [(a, b, c), (x, y, z)] is written with a double arc in the behavior graph of the source block of A and with a thick arc in the behavior graph of its target block. It can be followed by the 3DT-step $\xrightarrow{(a,b,c)}$, impacting only the target block of A. Dot symbols (•) are omitted. We denote by R, S, T, U, V, W some factors of the source and target blocks of A: the consequence of activating A is to allow U and V to be swapped in target(A).

3.2 Basic Blocks

We now define four basic blocks: copy, and, or, and var. They are studied independently in this section, before being assembled in Section 3.3. Each of these blocks is defined by a word and a set of triples. We distinguish internal triples, for which all three elements appear in a single block, from external triples, which are part of an input/output variable, and for which only one or two elements appear in the block. Note that each external triple is part of an input (resp. output) variable, which itself must be an output (resp. input) of another block, the other block containing the remaining elements of the triple.

We then draw the behavior graph of each of these blocks (Figures 9 to 12): in each case, we assume that the block is in a valid context, and follow exhaustively the 3DT-steps that can be applied on it. We then give another graph (Figures 13a to 13d), obtained from the behavior graph by contracting all arcs corresponding to 3DT-steps using internal triples, i.e. we assimilate every pair of nodes linked by such an arc. Hence, only the arcs corresponding to the activation of an input/output variable remain. From this second figure, we derive a property describing the behavior of the block, in terms of activating input and output variables (always provided this block is in a valid context). It must be kept in mind that for any variable, it is the state of the source block which determines whether it can be activated, whereas the activation itself affects mostly the target block.

3.2.1 The block copy

This block aims at duplicating a variable: any of the two output variables can only be activated after the input variable has been activated.

Input variable: A = [(a, b, c), (x, y, z)].

Output variables: $A_1 = [(a_1, b_1, c_1), (x_1, y_1, z_1)]$ and $A_2 = [(a_2, b_2, c_2), (x_2, y_2, z_2)]$.

Internal triple: (d, e, f).

Definition:

$$[A_1, A_2] = \mathsf{copy}(A) \hspace{0.5cm} = \hspace{0.5cm} \boxed{ a \; y_1 \; e \; z \; d \; y_2 \; x_1 \; b_1 \; c \; x_2 \; b_2 \; f }$$

Property 14. In a block $[A_1, A_2] = \text{copy}(A)$ in a valid context, the possible orders in which A, A_1 and A_2 can be activated are (A, A_1, A_2) and (A, A_2, A_1) .

Proof. See Figures 9 and 13a.

3.2.2 The block and

This block aims at simulating a conjunction: the output variable can only be activated after both input variables have been activated.

Input variables: $A_1 = [(a_1, b_1, c_1), (x_1, y_1, z_1)]$ and $A_2 = [(a_2, b_2, c_2), (x_2, y_2, z_2)].$

Output variable: A = [(a, b, c), (x, y, z)].

Internal triple: (d, e, f).

Definition:

Property 15. In a block $A = \text{and}(A_1, A_2)$ in a valid context, the possible orders in which A, A_1 and A_2 can be activated are (A_1, A_2, A) and (A_2, A_1, A) .

Proof. See Figures 10 and 13b.

3.2.3 The block or

This block aims at simulating a disjunction: the output variable can be activated as soon as any of the two input variables is activated.

Input variables: $A_1 = [(a_1, b_1, c_1), (x_1, y_1, z_1)]$ and $A_2 = [(a_2, b_2, c_2), (x_2, y_2, z_2)]$.

Output variable: A = [(a, b, c), (x, y, z)].

Internal triples: (a', b', c') and (d, e, f).

Definition:

Property 16. In a block $A = \text{or}(A_1, A_2)$ in a valid context, the possible orders in which A, A_1 and A_2 can be activated are (A_1, A, A_2) , (A_2, A, A_1) , (A_1, A_2, A) and (A_2, A_1, A) .

Proof. See Figures 11 and 13c.

3.2.4 The block var

This block aims at simulating a boolean variable: in a first stage, only one of the two output variables can be activated. The other needs the activation of the input variable to be activated. Input variable: A = [(a, b, c), (x, y, z)].

Output variables: $A_1 = [(a_1, b_1, c_1), (x_1, y_1, z_1)], A_2 = [(a_2, b_2, c_2), (x_2, y_2, z_2)].$

Internal triples: (d_1, e_1, f_1) , (d_2, e_2, f_2) and (a', b', c').

Definition:

$$[A_1, A_2] = \mathsf{var}(A) \qquad = \qquad \boxed{ d_1 \ y_1 \ a \ d_2 \ y_2 \ e_1 \ a' \ e_2 \ x_1 \ b_1 \ f_1 \ c' \ z \ b' \ c \ x_2 \ b_2 \ f_2 }$$

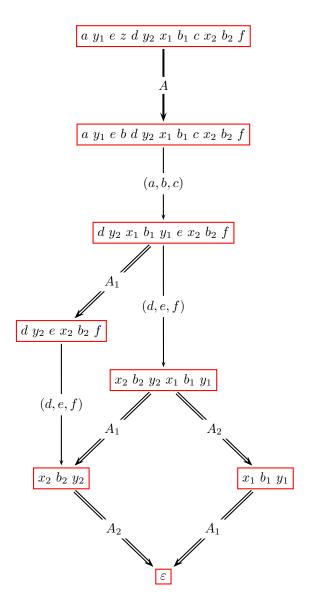


Figure 9: Behavior graph of the block $[A_1, A_2] = \mathsf{copy}(A)$. A thick (resp. double) arc corresponds to the 3DT-step $\xrightarrow{(x,y,z)}$ for an input (resp. output) variable [(a,b,c),(x,y,z)].

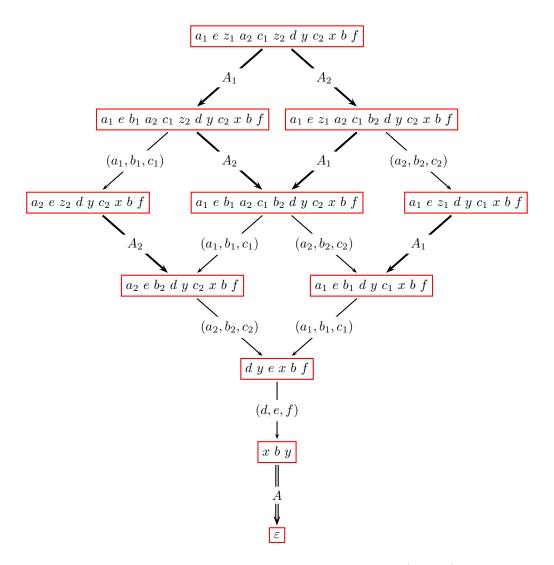


Figure 10: Behavior graph of the block $A = and(A_1, A_2)$.

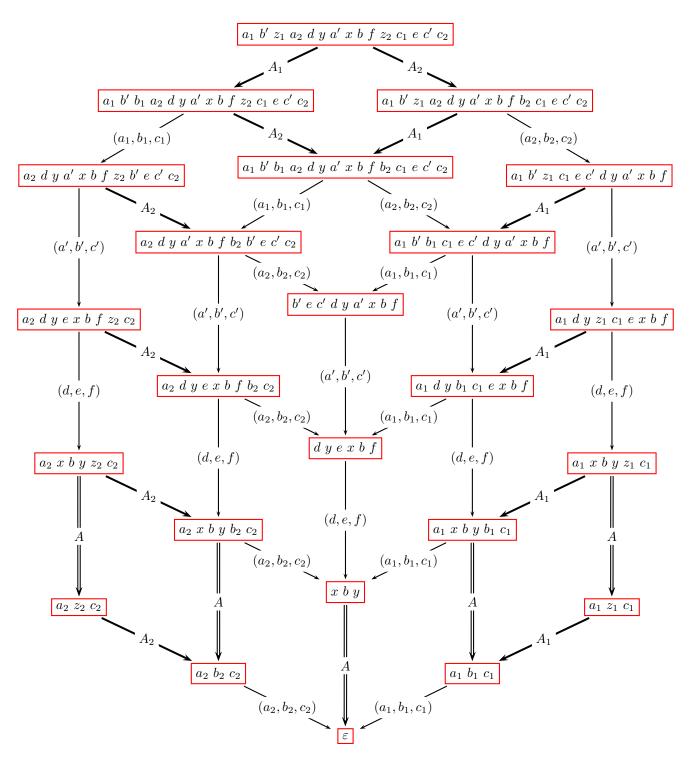


Figure 11: Behavior graph of the block $A = or(A_1, A_2)$.

Property 17. In a block $[A_1, A_2] = \text{var}(A)$ in a valid context, the possible orders in which A, A_1 and A_2 can be activated are (A_1, A, A_2) , (A_2, A, A_1) , (A, A_1, A_2) and (A, A_2, A_1) .

Proof. See Figures 12 and 13d. \Box

With such a block, if A is not activated first, one needs to make a choice between activating A_1 or A_2 . Once A is activated, however, all remaining output variables are activable.

3.2.5 Assembling the blocks copy, and, or, var.

Definition 14 (Assembling of basic blocks). An assembling of basic blocks (I, \mathcal{B}) is composed of a 3DT-instance I and an l-block-decomposition \mathcal{B} obtained by the following process:

- Create a set of variables A.
- Define I = ⟨Σ, T, ψ⟩ by its word representation, as a concatenation of l factors B₁^{*} B₂^{*} ... B_l^{*} and a set of triples T, where each B_h^{*} is one of the blocks [A₁, A₂] = copy(A), A = and(A₁, A₂), A = or(A₁, A₂) or [A₁, A₂] = var(A), with A₁, A₂, A ∈ A (such that each X ∈ A appears in the input of exactly one block, and in the output of exactly one other block); and where T is the union of the set of internal triples needed in each block, and the set of external triples defined by the variables of A.

Example 2.

We create a 3DT-instance I with a 2-block-decomposition \mathcal{B} such that (I,\mathcal{B}) is an assembling of basic blocks, defined as follows:

- I uses three variables, $\mathcal{A} = \{X_1, X_2, Y\}$
- the word representation of I is the concatenation of $[X_1, X_2] = \mathsf{var}(Y)$ and $Y = \mathsf{or}(X_1, X_2)$

With $X_1 = [(a_1, b_1, c_1), (x_1, y_1, z_1)]$, $X_2 = [(a_2, b_2, c_2), (x_2, y_2, z_2)]$, Y = [(a, b, c), (x, y, z)], and the internal triples $(d_1, e_1, f_1), (d_2, e_2, f_2), (a', b', c')$ for the block var, and (a'', b'', c''), (d, e, f) for the block or, the word representation of I is the following (note that its 2-block-decomposition is (0, 18)):

$$I = d_1 \ y_1 \ a \ d_2 \ y_2 \ e_1 \ a' \ e_2 \ x_1 \ b_1 \ f_1 \ c' \ z \ b' \ c \ x_2 \ b_2 \ f_2 \quad a_1 \ b'' \ z_1 \ a_2 \ d \ y \ a'' \ x \ b \ f \ z_2 \ c_1 \ e \ c'' \ c_2$$

Indeed, a possible sequence of 3DT-steps leading from I to ε is given in Figure 14, hence I is 3DT-collapsible.

Lemma 18. Let I' be a 3DT-instance with an l-block-decomposition \mathcal{B}' , such that (I', \mathcal{B}') is obtained from an assembling of basic blocks (I, \mathcal{B}) after any number of 3DT-steps, i.e. there exist $k \geq 0$ triples (d_i, e_i, f_i) , $i \in [1; k]$, such that $(I, \mathcal{B}) \xrightarrow{(d_1, e_1, f_1)} \cdots \xrightarrow{(d_k, e_k, f_k)} (I', \mathcal{B}')$.

Then (I', \mathcal{B}') is a valid context. Moreover, if the set of variables of (I', \mathcal{B}') is empty, then I' is 3DT-collapsible.

Proof. Write \mathcal{A} the set of variables used to define (I, \mathcal{B}) . We write $I = \langle \Sigma, T, \psi \rangle$ and $I' = \langle \Sigma', T', \psi' \rangle$. We prove that (I', \mathcal{B}') is a valid context by induction on k (the number of 3DT-steps between (I, \mathcal{B}) and (I', \mathcal{B}')). We also prove that for each $h \in [1; l]$, \mathcal{B}'_h appears as a node in the behavior graph of \mathcal{B}_h .

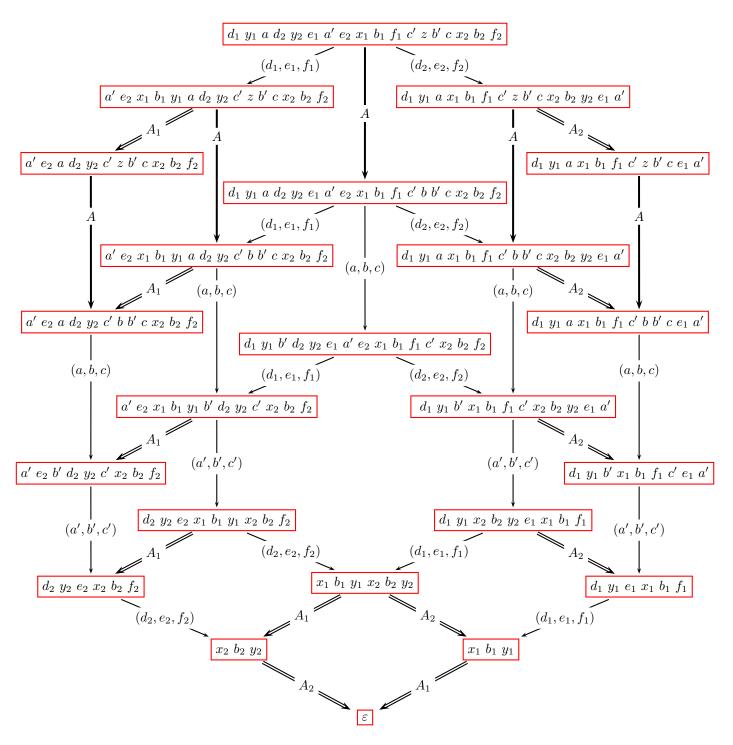
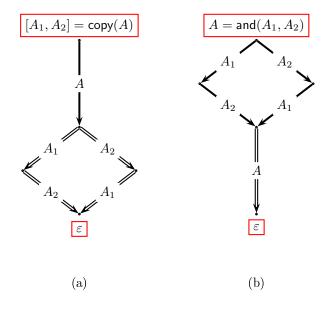


Figure 12: Behavior graph of the block $[A_1, A_2] = \text{var}(A)$.



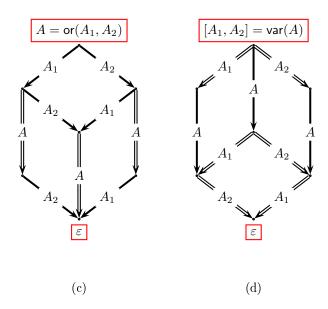


Figure 13: Abstract representations of the blocks copy, and, or, and var, obtained from each behavior graph (Figures 9, 10, 11 and 12) by contracting arcs corresponding to internal triples, and keeping only the arcs corresponding to variables. We see, for each block, which output variables are activable, depending on which variables have been activated.

Figure 14: 3DT-collapsibility of the assembling of basic blocks $[X_1, X_2] = \text{var}(Y)$ and $Y = \text{or}(X_1, X_2)$. For each 3DT-step, the three elements that are deleted from the alphabet are in bold, the elements that are swapped by the corresponding transposition are underlined. Vertical bars give the limits of the blocks in the 2-block-decomposition, and dot symbols are omitted.

Suppose first that k=0. We show that the set of external triples of $(I,\mathcal{B})=(I',\mathcal{B}')$ can be partitioned into valid variables, namely into \mathcal{A} . Indeed, from the definition of each block, for each $\sigma \in \Sigma$, σ is either part of an internal triple, or appears in a variable $A \in \mathcal{A}$. Conversely, for each $A = [(a,b,c),(x,y,z)] \in \mathcal{A}$, b, x and y appear in the block having A for output, and a, c and z appear in the block having A for input. Hence (a,b,c) and (x,y,z) are indeed two external triples of (I,\mathcal{B}) . Hence each variable satisfies conditions (i) and (ii) of Definition 12. Conditions (iii) and (iv) can be checked in the definition of each block: we have, for each output variable, $y \prec x$, and for each input variable, $a \prec z \prec c$. Finally, each \mathcal{B}_h appears in its own behavior graph.

Suppose now that (I', \mathcal{B}') is obtained from (I, \mathcal{B}) after k 3DT-steps, k > 0. Then there exists a 3DT-instance with an l-block-decomposition (I'', \mathcal{B}'') such that:

$$(I,\mathcal{B})\xrightarrow{(d_1,e_1,f_1)}\cdots\xrightarrow{(d_{k-1},e_{k-1},f_{k-1})}(I'',\mathcal{B}'')\xrightarrow{(d_k,e_k,f_k)}(I',\mathcal{B}').$$

Consider $h \in [1; l]$. By induction hypothesis, since \mathcal{B}''_h is in a valid context (I'', \mathcal{B}'') , then, depending on (d_k, e_k, f_k) , either $\mathcal{B}'_h = \mathcal{B}''_h$, either there is an arc from \mathcal{B}''_h to \mathcal{B}'_h in the behavior graph. Hence \mathcal{B}'_h is indeed a node in this graph. By Property 13, we know that the set of external triples of (I', \mathcal{B}') can be partitioned into variables satisfying conditions (i) and (ii) of Definition 12. Hence we need to prove that each variable satisfies conditions (iii) and (iv): we verify, for each node of each behavior graph, that $x \prec y \Rightarrow x \triangleleft b \triangleleft y$ (resp. $a \prec z \prec c$) for each output (resp. input) variable A = [(a, b, c), (x, y, z)] of the block. This achieves the induction proof.

We finally need to consider the case where the set of variables of (I', \mathcal{B}') is empty. Then for each $h \in [1; l]$ we either have $\mathcal{B}'_h = \varepsilon$, or $\mathcal{B}'_h = a_h \ b_h \ c_h$ for some internal triple (a_h, b_h, c_h) (in the case where \mathcal{B}_h is a block or). Then (I', \mathcal{B}') is indeed 3DT-collapsible: simply follow in any order the 3DT-step $\xrightarrow{(a_h, b_h, c_h)}$ for each remaining triple (a_h, b_h, c_h) .

3.3 Construction

Let ϕ be a boolean formula, over the boolean variables x_1, \ldots, x_m , given in conjunctive normal form: $\phi = C_1 \wedge C_2 \wedge \ldots \wedge C_{\gamma}$. Each clause C_c ($c \in [1; \gamma]$) is the disjunction of a number of literals, x_i or $\neg x_i$, $i \in [1; m]$. We write q_i (resp. \bar{q}_i) the number of occurrences of the literal x_i (resp. $\neg x_i$) in ϕ , $i \in [1; m]$. We also write $k(C_c)$ the number of literals appearing in the clause C_c , $c \in [1; \gamma]$. We can assume that $\gamma \geq 2$, that for each $c \in [1; \gamma]$, we have $k(C_c) \geq 2$, and that for each $i \in [1; m]$, $q_i \geq 2$ and $\bar{q}_i \geq 2$. (Otherwise, we can always add clauses of the form $(x_i \vee \neg x_i)$ to ϕ , or duplicate the literals appearing in the clauses C_c such that $k(C_c) = 1$.) In order to distinguish variables of an l-block-decomposition from x_1, \ldots, x_m , we always use the term boolean variable for the latter.

The 3DT-instance I_{ϕ} is defined as an assembling of basic blocks: we first define a set of variables, then we list the blocks of which the word representation of I_{ϕ} is the concatenation. It is necessary that each variable is part of the input (resp. the output) of exactly one block. Note that the relative order of the blocks is of no importance. We simply try, for readability reasons, to ensure that the source of a variable appears before its target, whenever possible. We say that a variable represents a term, i.e. a literal, clause or formula, if it can be activated only if this term is true (for some fixed assignment of the boolean variables), or if ϕ is satisfied by this assignment. We also say that a block defines a variable if it is its source block.

The construction of I_{ϕ} is done as follows (see Figure 15 for an example):

- Create a set of variables:
 - For each $i \in [1; m]$, create $q_i + 1$ variables representing x_i : X_i and X_i^j , $j \in [1; q_i]$, and $\bar{q}_i + 1$ variables representing $\neg x_i$: \bar{X}_i and \bar{X}_i^j , $j \in [1; \bar{q}_i]$.
 - For each $c \in [1; \gamma]$, create a variable Γ_c representing the clause C_c .
 - Create m+1 variables, A_{ϕ} and A_{ϕ}^{i} , $i \in [1; m]$, representing the formula ϕ . We will show that A_{ϕ} has a key role in the construction: it can be activated only if ϕ is satisfiable, and, once activated, it allows every remaining variable to be activated.
 - We also use a number of intermediate variables, with names U, \bar{U}, V, W and Y.
- Start with an empty 3DT-instance ε , and add blocks successively:
 - For each $i \in [1; m]$, add the following $q_i + \bar{q}_i 1$ blocks defining the variables X_i, X_i^j $(j \in [1; q_i])$, and \bar{X}_i, \bar{X}_i^j $(j \in [1; \bar{q}_i])$:

$$\begin{split} [X_i, \bar{X}_i] &= \mathsf{var}(A_\phi^i) \\ [X_i^1, U_i^2] &= \mathsf{copy}(X_i) \\ [X_i^2, U_i^3] &= \mathsf{copy}(U_i^2) \\ \vdots \\ [X_i^{q_i-2}, U_i^{q_i-1}] &= \mathsf{copy}(U_i^{q_i-2}) \\ [X_i^{q_i-2}, X_i^{q_i}] &= \mathsf{copy}(U_i^{q_i-2}) \\ [X_i^{q_i-1}, X_i^{q_i}] &= \mathsf{copy}(U_i^{q_i-1}) \\ [X_i^{\bar{q}_i-1}, \bar{X}_i^{\bar{q}_i}] &= \mathsf{copy}(\bar{U}_i^{\bar{q}_i-1}) \\ [\bar{X}_i^{\bar{q}_i-1}, \bar{X}_i^{\bar{q}_i}] &= \mathsf{copy}(\bar{U}_i^{\bar{q}_i-1}) \\ [\bar{X}_i^{\bar{q}_i-1}, \bar{X}_i^{\bar{q}_i}] &= \mathsf{copy}(\bar{U}_i^{\bar{q}_i-1}) \end{split}$$

- For each $c \in [1; \gamma]$, let $C_c = \lambda_1 \vee \lambda_2 \vee \ldots \vee \lambda_k$, with $k = k(C_c)$. Let each λ_p , $p \in [1; k]$, be the j-th occurrence of a literal x_i or $\neg x_i$, for some $i \in [1; m]$ and $j \in [1; q_i]$ (resp. $j \in [1; \bar{q}_i]$). We respectively write $L_p = X_i^j$ or $L_p = \bar{X}_i^j$. We add the following k-1 blocks defining Γ_c :

$$\begin{split} V_c^2 &= \operatorname{or}(L_1, L_2) \\ V_c^3 &= \operatorname{or}(V_c^2, L_3) \\ &\vdots \\ V_c^{k-1} &= \operatorname{or}(V_c^{k-2}, L_{k-1}) \\ \Gamma_c &= \operatorname{or}(V_c^{k-1}, L_k) \end{split}$$
 (**)

– Since $\phi = C_1 \wedge C_2 \wedge \ldots \wedge C_{\gamma}$, the formula variable A_{ϕ} is defined by the following $\gamma - 1$ blocks:

$$\begin{split} W_2 &= \mathsf{and}(\Gamma_1, \Gamma_2) \\ W_3 &= \mathsf{and}(W_2, \Gamma_3) \\ & \vdots \\ W_{\gamma-1} &= \mathsf{and}(W_{\gamma-2}, \Gamma_{\gamma-1}) \\ A_\phi &= \mathsf{and}(W_{\gamma-1}, \Gamma_l) \end{split}$$
 (***)

- The m copies $A_{\phi}^1, \ldots, A_{\phi}^m$ of A_{ϕ} are defined with the following m-1 blocks:

$$\begin{split} [A_{\phi}^{1},Y_{2}] &= \mathsf{copy}(A_{\phi}) \\ [A_{\phi}^{2},Y_{3}] &= \mathsf{copy}(Y_{2}) \\ & \vdots \\ [A_{\phi}^{m-2},Y_{m-1}] &= \mathsf{copy}(Y_{m-2}) \\ [A_{\phi}^{m-1},A_{\phi}^{m}] &= \mathsf{copy}(Y_{m-1}) \end{split}$$

3.4 Main Result

Theorem 19. Let ϕ be a boolean formula, and I_{ϕ} the 3DT-instance defined in Section 3.3. The construction of I_{ϕ} is polynomial in the size of ϕ , and ϕ is satisfiable iff I_{ϕ} is 3DT-collapsible.

Proof. The polynomial time complexity of the construction of I_{ϕ} is trivial. We use the same notations as in the construction, with \mathcal{B} the block decomposition of I_{ϕ} . One can easily check, in (*), (***), (****) and (*****), each variable has exactly one source block and one target block. Then, by Lemma 18, we know that (I_{ϕ}, \mathcal{B}) is a valid context, and remains so after any number of 3DT-steps, hence properties 14, 15, 16 and 17 are satisfied by respectively each block copy, and, or and var of I_{ϕ} .

 \implies Assume first that ϕ is satisfiable. Consider a truth assignment satisfying ϕ : let P be the set of indices $i \in [1; m]$ such that x_i is assigned to true. Starting from I_{ϕ} , we can follow a path of 3DT-steps that activates all the variables of I_{ϕ} in the following order:

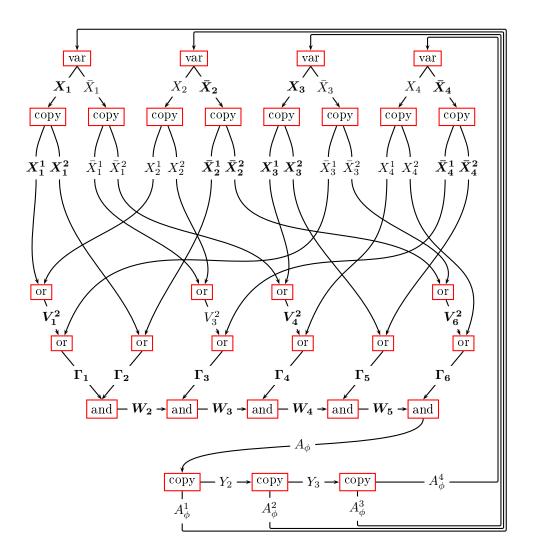


Figure 15: Schematic diagram of the blocks defining I_{ϕ} for $\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2 \vee \neg x_4) \wedge (\neg x_1 \vee x_3 \vee x_4) \wedge (\neg x_2 \vee \neg x_3 \vee x_4)$. For each variable, we draw an arc between its source and target block. Note that ϕ is satisfiable (e.g. with the assignment $x_1 = x_3 = true$ and $x_2 = x_4 = false$). A set of variables that can be activated before A_{ϕ} is in bold, they correspond to the terms being true in ϕ for the assignment $x_1 = x_3 = true$ and $x_2 = x_4 = false$.

- For $i \in [1; m]$, if $i \in P$, activate X_i in the corresponding block var in (*). Then, with the blocks copy, activate successively all intermediate variables U_i^j for j = 2 to $q_i 1$, and variables X_i^j for $j \in [1; q_i]$.
 - Otherwise, if $i \notin P$, activate \bar{X}_i , all intermediate variables \bar{U}_i^j for j = 2 to $\bar{q}_i 1$, and the variables \bar{X}_i^j for $j \in [1; \bar{q}_i]$
- For each $c \in [1; \gamma]$, let $C_c = \lambda_1 \vee \lambda_2 \vee \ldots \vee \lambda_k$, with $k = k(C_c)$. Since C_c is true with the selected truth assignment, at least one literal λ_{p_0} , $p_0 \in [1; k]$, is true. If λ_{p_0} is the j-th occurrence of a literal x_i or $\neg x_i$, then the corresponding variable L_{p_0} ($L_{p_0} = X_i^j$ or $L_{p_0} = \bar{X}_i^j$) has been activated previously. Using the blocks or in (**), we activate successively each intermediate variable V_c^p for $p = p_0$ to p = k 1, and finally we activate the variable Γ_c .
- Since all variables Γ_c , $c \in [1; \gamma]$, have been activated, using the blocks and in (***), we activate each intermediate variable W_c for c = 2 to $c = \gamma 1$, and the formula variable A_{ϕ} .
- With the blocks copy in (****), we activate successively all the intermediate variables Y_i , $i \in [2; m-1]$ and the m copies $A_{\phi}^1, \ldots, A_{\phi}^m$ of A_{ϕ} .
- For $i \in [1; m]$, since the variable A^i_{ϕ} has been activated, we activate in the block var of (*) the remaining variable X_i or \bar{X}_i . We also activate all its copies and corresponding intermediate variables U^j_i or \bar{U}^j_i .
- For $c \in [1; \gamma]$, in (**), since all variables L_p have been activated, we activate the remaining intermediate variables V_c^p .
- At this point every variable has been activated. Using again Lemma 18, we know that the resulting instance is 3DT-collapsible, and can be reduced down to the empty 3DT-instance ε .

Hence I_{ϕ} is 3DT-collapsible.

- Æ Assume now that I_{ϕ} is 3DT-collapsible: we consider a sequence of 3DT-steps reducing I_{ϕ} to ε . This sequence gives a total order on the set of variables: the order in which they are activated. We write Q the set of variables activated before A_{ϕ} , and $P \subseteq [1; m]$ the set of indices i such that $X_i \in Q$ (see the variables in bold in Figure 15). We show that the truth assignment defined by $(x_i = true \Leftrightarrow i \in P)$ satisfies the formula ϕ .
 - For each $i \in [1; m]$, A^i_{ϕ} cannot belong to Q, using the property of the block copy in (****) (each A^i_{ϕ} can only be activated after A_{ϕ}). Hence, with the block var in (*), we have $\bar{X}_i \in Q \Rightarrow X_i \notin Q$. Moreover, with the block copy, we have

$$\forall 1 \le j \le q_i, \quad X_i^j \in Q \Rightarrow X_i \in Q \tag{a}$$

$$\forall 1 \le j \le \bar{q}_i, \quad \bar{X}_i^j \in Q \Rightarrow \bar{X}_i \in Q \Rightarrow X_i \notin Q \tag{b}$$

• Since A_{ϕ} is defined in a block $A_{\phi} = \operatorname{and}(W_{\gamma-1}, \Gamma_{\gamma})$ in (***), we necessarily have $W_{\gamma-1} \in Q$ and $\Gamma_{\gamma} \in Q$. Likewise, since $W_{\gamma-1}$ is defined by $W_{\gamma-1} = \operatorname{and}(W_{\gamma-2}, \Gamma_{\gamma-1})$, we also have $W_{\gamma-2} \in Q$ and $\Gamma_{\gamma-1} \in Q$. Applying this reasoning recursively, we have $\Gamma_c \in Q$ for each $c \in [1; \gamma]$.

• For each $c \in [1; \gamma]$, consider the clause $C_c = \lambda_1 \vee \lambda_2 \vee \ldots \vee \lambda_k$, with $k = k(C_c)$. Using the property of the block or in (**), there exists some $p_0 \in [1; k]$ such that the variable L_{p_0} is activated before Γ_c : hence $L_{p_0} \in Q$. If the corresponding literal λ_{p_0} is the j-th occurrence of x_i (respectively, $\neg x_i$), then $L_{p_0} = X_i^j$ (resp., $L_{p_0} = \bar{X}_i^j$), thus by (a) (resp. (b)), $X_i \in Q$ (resp., $X_i \notin Q$), and consequently $i \in P$ (resp., $i \notin P$). In both cases, the literal λ_{p_0} is true in the truth assignment defined by $(x_i = true \Leftrightarrow i \in P)$.

If I_{ϕ} is 3DT-collapsible, we have found a truth assignment such that at least one literal is true in each clause of the formula ϕ , and thus ϕ is satisfiable.

4 Sorting by Transpositions is NP-Hard

As noted previously, there is no guarantee that any 3DT-instance I has an equivalent permutation π . However, with the following theorem, we show that such a permutation can be found in the special case of assemblings of basic blocks, which is the case we are interested in, in order to complete our reduction.

Theorem 20. Let I be a 3DT-instance of span n with \mathcal{B} an l-block-decomposition such that (I,\mathcal{B}) is an assembling of basic blocks. Then there exists a permutation π_I , computable in polynomial time in n, such that $I \sim \pi_I$.

An example of the construction of π_I for the 3DT-instance defined in Example 2 is given in Figure 16.

Proof. Let \mathcal{A} be the set of variables of the l-block-decomposition \mathcal{B} of $I = \langle \Sigma, T, \psi \rangle$. Let n be the span of I, and L its domain. Note that L = [1; n]. For any $h \in [1; l]$, we write $ni(\mathcal{B}_h)$ (resp. $no(\mathcal{B}_h)$) the number of input (resp. output) variables of \mathcal{B}_h . We also define two integers p_h, q_h by:

$$p_1 = 0$$

$$\forall h \in [1; l], \qquad q_h = p_h + t_h - s_h + 3(ni(\mathcal{B}_h) - no(\mathcal{B}_h))$$

$$\forall h \in [2; l], \qquad p_h = q_{h-1}$$

The permutation π_I will be defined such that p_h and q_h have the following property for any $h \in [1; l]$: $\pi_I(s_h) = p_h$, and $\pi_I(t_h) = q_h$.

We also define two applications α, β over the set \mathcal{A} of variables. The permutation π_I will be defined so that, for any variable A = [(a, b, c), (x, y, z)], we have $\pi_I(\psi(a) - 1) = \alpha(A)$ and $\pi_I(\psi(z) - 1) = \beta(A)$. In order to have this property, α and β are defined as follows.

For each $h \in [1; l]$:

• If \mathcal{B}_h is a block of the kind $[A_1, A_2] = \mathsf{copy}(A)$, define

$$\alpha(A) = p_h, \ \beta(A) = p_h + 4.$$

• If \mathcal{B}_h is a block of the kind $A = \operatorname{and}(A_1, A_2)$, define

$$\alpha(A_1) = p_h$$
, $\beta(A_1) = p_h + 7$, $\alpha(A_2) = p_h + 3$, $\beta(A_2) = p_h + 9$.

• If \mathcal{B}_h is a block of the kind $A = \text{or}(A_1, A_2)$, define

$$\alpha(A_1) = p_h$$
, $\beta(A_1) = p_h + 13$, $\alpha(A_2) = p_h + 3$, $\beta(A_2) = p_h + 16$.

• If \mathcal{B}_h is a block of the kind $[A_1, A_2] = \mathsf{var}(A)$, define

$$\alpha(A) = p_b + 5, \ \beta(A) = p_b + 9.$$

Note that for every $A \in \mathcal{A}$, $\alpha(A)$ and $\beta(A)$ are defined once and only once, depending on the kind of the block $\mathcal{B}_{target(A)}$. The permutation π_I is designed in such a way that the image by π_I of an interval $[s_h + 1; t_h]$ is essentially the interval $[p_h + 1; q_h]$. However, there are exceptions:

namely, for each variable A, the integers $\alpha(A) + 1$, $\alpha(A) + 2$, $\beta(A) + 1$, which are included in $[\![p_{target(A)} + 1; q_{target(A)}]\!]$, are in the image of $[\![s_{source(A)} + 1; t_{source(A)}]\!]$. This is formally described as follows. For each $h \in [\![1; k]\!]$ we define a set P_h by:

$$P_{h} = \llbracket p_{h} + 1 \; ; \; q_{h} \rrbracket \quad \cup \quad \bigcup_{\substack{A \text{ output of } \mathcal{B}_{h}}} \{\alpha(A) + 1, \alpha(A) + 2, \beta(A) + 1\}$$
$$- \bigcup_{\substack{A \text{ input of } \mathcal{B}_{h}}} \{\alpha(A) + 1, \alpha(A) + 2, \beta(A) + 1\}$$

We note that the sets $\{\alpha(A) + 1, \alpha(A) + 2, \beta(A) + 1\}$ are distinct for different variables A, and are each included in their respective interval $[p_{target(A)} + 1; q_{target(A)}]$. Hence for any $h \in [1; l]$, we have $|P_h| = q_h - p_h + 3no(\mathcal{B}_h) - 3ni(\mathcal{B}_h) = t_h - s_h$. Moreover, the sets P_h , $h \in [1; l]$, form a partition of the set [1; n].

We can now create the permutation π_I . The image of 0 is 0, and for each h_0 from 1 to l, we define the restriction of π_I over $[s_{h_0} + 1; t_{h_0}]$ as a permutation of P_{h_0} , with the constraint that $\pi_I(t_{h_0}) = q_{h_0}$. Note that, if this condition is fulfilled, then we can assume $\pi_I(s_{h_0}) = p_{h_0}$, since, if $h_0 = 1$, $\pi_I(s_1) = \pi_I(0) = 0 = p_1$, and if $h_0 > 1$, $\pi_I(s_{h_0}) = \pi_I(t_{h_0-1}) = q_{h_0-1} = p_{h_0}$.

The definition of π_I over each kind of block is given in Table 1. This table is obtained by applying the following rules, until $\pi_I(u)$ is defined for all $u \in [s_{h_0} + 1; t_{h_0}]$.

$$\forall A = [(a, b, c), (x, y, z)] \text{ input variable of } \mathcal{B}_{h_0}$$
$$\pi_I(\psi(z)) = \alpha(A) + 3 \tag{R_1}$$

$$\pi_I(\psi(c)) = \beta(A) + 2 \tag{R_2}$$

 $\forall A = [(a,b,c),(x,y,z)]$ output variable of \mathcal{B}_{h_0}

$$\pi_I(\psi(x)) = \beta(A) + 1 \tag{R_3}$$

$$\pi_I(\psi(b)) = \alpha(A) + 1 \tag{R_4}$$

 $\forall u \in [\![s_{h_0}+1\,;\,t_{h_0}]\!] \text{ such that } succ_{I_\phi}^{-1}(u) \in [\![s_{h_0}+1\,;\,t_{h_0}]\!]$

$$\pi_I(u) = \pi_I(succ_I^{-1}(u) - 1) + 1$$
 (R₅)

We can see in Table 1 that rules (R_1) and (R_2) indeed apply to every input variable, and rules (R_3) and (R_4) apply to every output variable. Moreover:

Rule
$$(R_5)$$
 applies to every $u \in \llbracket s_{h_0+1} ; t_{h_0} \rrbracket$ such that $u \notin \{\psi(b), \psi(c), \psi(x), \psi(z) \mid A = [(a, b, c), (x, y, z)] \text{ input/output of } \mathcal{B}_{h_0} \}.$ (P_1)

A simple case by case analysis shows that the following properties are also satisfied.

$$\pi_I$$
 defines a bijection from $\llbracket s_{h_0} + 1; t_{h_0} \rrbracket$ to P_{h_0} such that $\pi_I(t_{h_0}) = q_{h_0}$ (P₂)

Table 1: Definition of π_I over an interval $[s_{h_0} + 1; t_{h_0}]$, where \mathcal{B}_{h_0} is one of the blocks copy, and, or, var. We write $s = s_{h_0}$ and $p = p_{h_0}$. We give the line $\psi^{-1}(u)$ as a reminder of the definition of each block. We also add a column for u = s as a reminder of the fact that $\pi_I(s) = p$.

• If \mathcal{B}_{h_0} is a block of the kind $[A_1, A_2] = \mathsf{copy}(A)$, we write $\alpha_1, \beta_1, \alpha_2, \beta_2$ the respective values of $\alpha(A_1), \beta(A_1), \alpha(A_2), \beta(A_2)$.

• If \mathcal{B}_{h_0} is a block of the kind $A = \operatorname{and}(A_1, A_2)$, we write α, β the respective values of $\alpha(A), \beta(A)$.

$$u = s$$
 $s+1$ $s+2$ $s+3$ $s+4$ $s+5$ $s+6$ $s+7$ $s+8$ $s+9$ $s+10$ $s+11$ $s+12$ $\pi_I(u) = p$ $p+14$ $p+7$ $p+3$ $p+13$ $p+9$ $p+6$ $\alpha+2$ $p+12$ $p+11$ $\beta+1$ $\alpha+1$ $p+15$ $\psi^{-1}(u) = a_1$ e z_1 a_2 c_1 z_2 d y c_2 x b f

• If \mathcal{B}_{h_0} is a block of the kind $A = \text{or}(A_1, A_2)$, we write α, β the respective values of $\alpha(A), \beta(A)$.

• If \mathcal{B}_{h_0} is a block of the kind $[A_1, A_2] = \mathsf{var}(A)$, we write $\alpha_1, \beta_1, \alpha_2, \beta_2$ the respective values of $\alpha(A_1), \beta(A_1), \alpha(A_2), \beta(A_2)$.

$$\forall A = [(a, b, c), (x, y, z)]$$
 input variable of \mathcal{B}_{h_0} ,

$$\pi_I(\psi(a) - 1) = \alpha(A) \tag{P_3}$$

$$\pi_I(\psi(z) - 1) = \beta(A) \tag{P_4}$$

 $\forall A = [(a, b, c), (x, y, z)]$ output variable of \mathcal{B}_{h_0} ,

$$\pi_I(\psi(y) - 1) = \alpha(A) + 2 \tag{P_5}$$

$$\pi_I(\psi(b) - 1) = \beta(A) + 1$$
 (P₆)

Now that we have defined the permutation π_I , we need to show that π_I is equivalent to I. Following Definition 10, we have $\pi_I(0) = 0$. Then, L = [1; n], so let us fix any $u \in [1; n]$, and verify that $\pi_I(u) = \pi_I(succ_I^{-1}(u) - 1) + 1$. Let h be the integer such that $u \in [s_h + 1; t_h]$.

First consider the most general case, where there is no variable A = [(a, b, c), (x, y, z)] such that $u \in \{\psi(b), \psi(c), \psi(x), \psi(z)\}$. Note that this case includes $u = \psi(d)$, where d is part of any internal triple. Then, by Property (P_1) , we know that Rule (R_5) applies to u, hence we directly have $\pi_I(u) = \pi_I(succ_I^{-1}(u) - 1) + 1$.

Suppose now that, for some variable A = [(a, b, c), (x, y, z)], we have $u \in \{\psi(b), \psi(c), \psi(x), \psi(z)\}$. Then Rules (R_1) and (R_2) , and Properties (P_3) and (P_4) apply in the target block of A. Also, Rules (R_3) and (R_4) , and Properties (P_5) and (P_6) apply in the source block of A. Combining all these equations together, we have:

$$\pi_I(\psi(b)) = \alpha(A) + 1 = \pi_I(\psi(a) - 1) + 1$$
by (R_3) and (P_3)
 $\pi_I(\psi(c)) = \beta(A) + 2 = \pi_I(\psi(b) - 1) + 1$
by (R_2) and (P_5)
 $\pi_I(\psi(x)) = \beta(A) + 1 = \pi_I(\psi(z) - 1) + 1$
by (R_4) and (P_4)
 $\pi_I(\psi(z)) = \alpha(A) + 3 = \pi_I(\psi(y) - 1) + 1$
by (R_1) and (R_2)

For $u = \psi(b)$ (resp. $\psi(c), \psi(x), \psi(z)$), we have $succ_I^{-1}(u) = \psi(a)$ (resp. $\psi(b), \psi(z), \psi(y)$). Hence, in all four cases, we have $\pi_I(u) = \pi_I(succ_I^{-1}(u) - 1) + 1$, which completes the proof that π_I is equivalent to I.

With the previous theorem, we now have all the necessary ingredients to prove the main result of this paper.

Theorem 21. The Sorting by Transpositions problem is NP-hard.

Proof. The reduction from SAT is as follows: given any instance ϕ of SAT, create a 3DT-instance I_{ϕ} , being an assembling of basic blocks, which is 3DT-collapsible iff ϕ is satisfiable (Theorem 19). Then create a 3-permutation $\pi_{I_{\phi}}$ equivalent to I_{ϕ} (Theorem 20). The above two steps can be done in polynomial time. Finally, set $k = d_b(\pi_{I_{\phi}})/3 = n/3$. We then have:

$$\phi$$
 is satisfiable $\Leftrightarrow I_{\phi}$ is 3DT-collapsible
$$\Leftrightarrow d_t(\pi_{I_{\phi}}) = k \text{ (by Theorem 9, since } \pi_{I_{\phi}} \sim I_{\phi})$$
$$\Leftrightarrow d_t(\pi_{I_{\phi}}) \leq k \text{ (by Property 4)}.$$

$[X_1,X_2]=var(Y)$	$Y = or(X_1, X_2)$
Input variable (target of): Y Output variables (source of): X_1, X_2	Input variables (target of): X_1, X_2 Output variable (source of): Y
$s_1 = 0$ $t_1 = 18$ $p_1 = 0$ $q_1 = p_1 + 18 - 3 = 15$	$s_2 = 18$ $t_2 = 33$ $p_2 = 15$ $q_2 = p_2 + 15 + 3 = 33$
$\alpha(Y) = p_1 + 5 = 5 \beta(Y) = p_1 + 9 = 9$	$\alpha(X_1) = p_2 = 15$ $\beta(X_1) = p_2 + 13 = 28$ $\alpha(X_2) = p_2 + 3 = 18$ $\beta(X_2) = p_2 + 16 = 31$
Definition of π_I over $[s_1 + 1; t_1]$:	Definition of π_I over $[s_2 + 1; t_2]$:
$u = 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9 \mid 10 \mid 11 \mid 12 \mid 13 \mid 14 \mid 15 \mid 16 \mid 17 \mid 18$	19 20 21 22 23 24 25 26 27 28 29 30 31 32 33
$\pi_I(u) = 0$ 17 5 3 20 12 1 14 4 29 16 13 9 8 2 11 32 19 15	22 28 18 24 7 27 26 10 6 31 21 30 25 23 33
$\psi^{-1}(u) = d_1 y_1 a d_2 y_2 e_1 a' e_2 x_1 b_1 f_1 c' z b' c x_2 b_2 f_2$	$\begin{bmatrix} a_1 \ b'' \ z_1 \ a_2 \ d \ y \ a'' \ x \ b \ f \ z_2 \ c_1 \ e \ c'' \ c_2 \end{bmatrix}$

Figure 16: Creation of a permutation π_I equivalent to the assembling of basic blocks $I = \langle \Sigma, T, \psi \rangle$ of span 33 defined in Example 2, following the proof of Theorem 20.

Note that the permutation π_I defined by Theorem 20 is in fact a 3-permutation, i.e. a permutation whose cycle graph contains only 3-cycles [3] (which is equivalent to saying that the application succ defined by $succ(u) = \pi_I^{-1}(\pi_I(u-1)+1)$ has no fixed point, and is such that $succ \circ succ \circ succ$ is the identity). Moreover, the number of breakpoints of π_I is $d_b(\pi_I) = n$. Hence we have the following corollary.

Corollary 22. The following two decision problems [8] are NP-hard:

- Given a permutation π of [0; n], is the equality $d_t(\pi) = d_b(\pi)/3$ satisfied?
- Given a 3-permutation π of [0; n], is the equality $d_t(\pi) = n/3$ satisfied?

Conclusion

In this paper we have proved that the SORTING BY TRANSPOSITIONS problem is NP-hard, thus answering a long-standing question. However, a number of questions remain open. For instance, does this problem admit a polynomial time approximation scheme? We note that the reduction we have provided does not answer this question, since it is not a linear reduction. Indeed, by our reduction, if a formula ϕ is not satisfiable, it can be seen that we have $d_t(\pi_{I_{\phi}}) = d_b(\pi_{I_{\phi}})/3 + 1$.

Also, does there exist some relevant parameters for which the problem is fixed parameter tractable? A parameter that comes to mind when dealing with the transposition distance is the size of the factors exchanged (e.g., the value $\max\{j-i,k-j\}$ for a transposition $\tau_{i,j,k}$). Does the problem become tractable if we bound this parameter? In fact, the answer to this question is no if we bound only the size of the smallest factor, $\min\{j-i,k-j\}$: in our reduction, this parameter is upper bounded by 6 for every transposition needed to sort $\pi_{I_{\phi}}$, independently of the formula ϕ .

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